G-STABLE PIECES AND LUSZTIG'S DIMENSION ESTIMATES

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Abstract. We use G-stable pieces to construct some equidimensional varieties and as a consequence, obtain Lusztig's dimension estimates [L2, section 4]. This is a generalization of [HL].

In Lemma 1.1 and Proposition 1.2, we assume that G is arbitrary connected algebraic group and $\tilde{G}$ is an algebraic group with identity component G.

Lemma 1.1. *Let $g \in \tilde{G}$. Define $i : \tilde{G} \to \tilde{G}$ by $i(h) = ghg^{-1}h^{-1}$. For any closed subgroup A of Z_G with $gAg^{-1} = A$, set $H_A = \{h \in G; i(h) \in A\}$. Then*

(1) H_A is an algebraic group and $i|_{H_A} : H_A \to A$ is a morphism of algebraic groups.

(2) $i(A)^0 = i(H_A)^0$.

(3) $\dim(H_A) = \dim(Z_G(g)) + \dim(A) - \dim(Z_A(g))$.

If $h, h' \in H_A$, then

$$\begin{aligned} i(hh') &= ghh'g^{-1}(h')^{-1}h^{-1} = (ghg^{-1}h^{-1})h(gh'g^{-1}(h')^{-1})h^{-1} \\ &= i(h)hi(h')h^{-1} = i(h)i(h') \in A \end{aligned}$$

and $hh' \in H_A$. If $h \in H_A$, then $i(h^{-1}) = h^{-1}i(h)^{-1}h = i(h)^{-1} \in A$ and $h^{-1} \in H_A$. Part (1) is proved.

Now $i(A)^0$ is a connected subgroup of $i(H_A)$. Define $\delta : A \to A$ by $\delta(z) = gzg^{-1}$. Then

$$\dim(i(A)) = \dim(A) - \dim(A^\delta).$$

Define $\sigma : A \to A$ by $\sigma(z) = \delta^{m-1}(z)\delta^{m-2}(z) \cdots z$, where m is the order of the automorphism δ. Then σ is a group homomorphism and

$$i(H_A) \subset \{z \in Z; \sigma(z) = 1\}.$$

Notice that $\sigma(A^\delta) = \{t^m; t \in A^\delta\}$ is of dimension $\dim(A^\delta)$. Thus

$$\begin{aligned} \dim(i(H_A)) &\leqslant \dim(A) - \dim(\sigma(A)) \leqslant \dim(A) - \dim(\sigma(A^\delta)) \\ &= \dim(A) - \dim(A^\delta). \end{aligned}$$

Therefore, $\dim(i(A)) = \dim(i(H_A)) = \dim(A) - \dim(A^\delta)$. Part (2) is proved.

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Since the kernal of $i|_{H_A}$ is $Z_G(g)$,

$$\begin{aligned}\dim(H_A) &= \dim(Z_G(g)) + \dim(i(H_A)) \\ &= \dim(Z_G(g)) + \dim(A) - \dim(A^\delta).\end{aligned}$$

Part (3) is proved. □

Proposition 1.2. *Let D, D' be connected group of $\tilde{G}$. Define $\delta : Z_G \to Z_G$ by $\delta(z) = gzg1$ for any $g \in D$ and $\delta' : Z_G \to Z_G$ by $\delta'(z) = g'z(g')1$ for any $g' \in D'$. Let c be a G-conjugacy class in D and Z be a closed subgroup of Z_G with $\delta(Z) = Z$. Set*

$$X = \{(g, g'); g \in cZ, g' \in D', gg'g1(g')1 \in Z\}.$$

If $X \neq \emptyset$, then $DD' = D'D$ and $\delta\delta'(z) = \delta'\delta(z)$ for $z \in Z_G$. Moreover, X is of pure dimension $\dim(G) + \dim(Z) - \dim(\frac{\delta'(Z)}{Z \cap \delta'(Z)})^\delta$.

It is easy to see that if $X \neq \emptyset$, then $DD' = D'D$. Thus for $g \in D$ and $g' \in D'$, $g1(g')1gg' \in G$ and $\delta1(\delta')1\delta\delta'(z) = z$ for all $z \in Z_G$.

Consider the projection map $X \to D$ defined by $(g, g') \mapsto g$. Let g be in the image and X_g be the fiber over g. Fix $g' \in X_g$. Set $i : \tilde{G} \to \tilde{G}$ by $i(h) = ghg1h1$. Then $i(hg') = i(h)i(g')$ for $h \in G$. Hence $X_g = H_Z g'$.

Let $z \in Z$. Then for $h \in G$,

$$(zg)(hg')(zg)1(hg')1 = zi(h)i(g')\delta'(z)1.$$

If $zi(h)i(g')\delta'(z)1 \in Z$, then $i(h) \in Z\delta'(Z)$. Hence $h \in H_{Z\delta'(Z)}$ and $\delta'(z) \in i(H_{Z\delta'(Z)})Z$. On the other hand, if $z \in Z$ with $\delta'(z) \in i(H_{Z\delta'(Z)})Z$, then there exists $h \in G$ with $zi(h)i(g')\delta'(z)1 \in Z$. Therefore, for $z \in Z$, $X_{zg} \neq \emptyset$ if and only if $z \in Z'$, where $Z' = \{z \in Z; \delta'(z) \in i(H_{Z\delta'(Z)})Z\}$. It is easy to see that Z' is an algebraic group.

By part (2) of the previous lemma, $i(H_{Z\delta'(Z)})^0 \subset i(Z\delta'(Z))$. Hence $\delta'(Z')^0 = \big(\delta'(Z) \cap i(Z\delta'(Z))Z\big)^0 = \big(\delta'(Z) \cap i(\delta'(Z))Z\big)^0$. Notice that $i(Z) \subset \delta(Z)Z = Z$ and $i\delta'(Z) \subset \delta\delta'(Z)\delta'(Z) = \delta'(Z)$. Now i induces a group morphism $\bar{i} : \delta'(Z)/(Z \cap \delta'(Z)) \to \delta'(Z)/(Z \cap \delta'(Z))$. We have that $z \in \delta'(Z) \cap i(\delta'(Z))Z$ if and only if $z(Z \cap \delta'(Z))$ is contained in the image of $\bar{i}$. Hence $\dim(Z') = \dim(\frac{\delta'(Z)}{Z \cap \delta'(Z)}) - \dim(\frac{\delta'(Z)}{Z \cap \delta'(Z)})^\delta + \dim(Z \cap \delta'(Z)) = \dim(Z) - \dim(\frac{\delta'(Z)}{Z \cap \delta'(Z)})^\delta$.

Set $Y = \{(h, z); z \in Z', h \in X_{zg}\}$. Then we have the projection map $Y \to Z'$ and each fiber is isomorphic to H_Z. Hence Y is of pure dimension $\dim(Z') + \dim(H_Z)$.

Consider the morphism $G \times Y \to X$ defined by

$$(g_1, h, z) \mapsto (g_1 h g_1 1, z g_1 g g_1 1).$$

Then this morphism is surjective and the fiber over (h, zg) is

$$\{(g_1, g_1 1 h g_1, i(g_1)z); g_1 \in H_Z\},$$

which is of pure dimension $\dim(H_Z)$. Therefore X is of pure dimension $\dim(G) + \dim(Z')$. □

1.3. From now on, we assume that G be a simply-connected, semisimple algebraic group over an algebraically closed field $\mathbf{k}$. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. Let I be the set of simple roots determined by B and T.

For any $J \subset I$, let P_J be the standard parabolic subgroup corresponding to J and $\mathcal{P}_J$ be the set of parabolic subgroups that are G-conjugate to P_J. We simply write $\mathcal{P}_\emptyset$ as $\mathcal{B}$. Let L_J be the Levi subgroup of P_J that contains T.

For any parabolic subgroup P, let U_P be the unipotent radical of P and H_P be the inverse image of the connected center of P/U_P under the projection map $\pi_P : P \to P/U_P$. We simply write U for U_B.

For $J \subset I$, we denote by W_J the standard parabolic subgroup of W generated by J and by W^J (resp. ${}^J W$) the set of minimal coset representatives in W/W_J (resp. $W_J \backslash W$). For $J, K \subset I$, we simply write $W^J \cap {}^K W$ as ${}^K W^J$.

For $P \in \mathcal{P}_J$ and $Q \in \mathcal{P}_K$, we write $\mathrm{pos}(P, Q) = w$ if $w \in {}^J W^K$ and there exists $g \in G$ such that $P = gP_J g^{-1}$, $Q = g\dot{w}P_K\dot{w}^{-1}g^{-1}$, where $\dot{w}$ is a representative of w in $N(T)$.

For $g \in G$ and $H \subset G$, we write ${}^g H$ for gHg^{-1}.

For any algebraic group H, let H^0 be its identity component.

1.4. Let σ be a diagram automorphism of G, i.e., an automorphism of G that stabilizes B and T and the order of σ as an automorphism of G coincides with the order of the associated permutation on I. We use the same symbol σ for the associated automorphism on W and associated permutation on I. Set $\tilde{G} = G \rtimes < \sigma >$, where $< \sigma >$ is the finite subgroup of G generated by σ. We simply write the element $(g, \sigma^n) \in \tilde{G}$ as $g\sigma^n$. For each element $g \in \tilde{G}$, we write g_s for its semisimple part and g_u its unipotent part.

Let $D = (G, \sigma)$ be a connected component of $\tilde{G}$. We have the following result.

Proposition 1.5. *Let $g \in D$. Then g is G^0-conjugate to an element of the form $t\sigma u$, where $t \in (T^\sigma)^0$ and u is a unipotent element in $Z^0_G(t\sigma_s)$.*

By [St, Lemma 7.3], after conjugate by an element in G^0, we may assume that $g \in B\sigma \subset B \ltimes < \sigma >$. Then $g_s \in B\sigma_s$ and $g_u \in B\sigma_u$. Then after conjugate by an element in B, we may assume that $g_s = t_1\sigma_s$ and $g_u = t_2\sigma_u u$ for $t_1, t_2 \in T$ and $u \in U$. By [L2, 1.2], after conjugate by an element in T, we may assume furthermore that $t_2 \in (T^{\sigma_u})^0$. Consider the group morphism $B \ltimes < \sigma > \to T \ltimes < \sigma >$. Since g_u is unipotent, then $t_2\sigma_u$ is also unipotent. Notice that t_2 commutes with σ_u. Then t_2 is unipotent and $t_2 = 1$.

Since σ is a diagram automorphism, σ_s and σ_u are also diagram automorphisms. In particular, $\rho^\vee(t) \in T^{\sigma_\sigma} \cap T^{\sigma_u}$ for all $t \in \mathbf{k}^\times$. Hence $\mathrm{Ad}(\rho^\vee(t))g_u \in Z_G(g_s)$. Since σ_u is contained in the closure

of $\{\mathrm{Ad}(\rho^\vee(t))g_u)\}$ and $Z_G(g_s)$ is closed, we have that $\sigma_u \in Z_G(g_s)$. We also have that $u \in Z_G^0(g_s)$.

Now σ_u commutes with $t_1\sigma_s$. Hence $t_1 \in T^{\sigma_u}$ By [Bo, 9.6], (T^{σ_u}) is connected. Notice that σ_s is an automorphism on T^{σ_u}. Then by [L2, 1.2], after conjugate by T^{σ_u}, $t_1 \in ((T^{\sigma_u})^{\sigma_s})^0 \subset (T^\sigma)^0$. □

1.6. Let $D//G$ be set of closed G-conjugacy classes in D. By geometric invariant theory, $D//G$ has a natural structure of affine variety and there is a well-defined morphism $\mathrm{St} : D \to D//G$ which maps the element $g \in D$ to the unique closed G-conjugacy class in D that is contained in the closure of the G-conjugacy class of g. If σ is trivial, then St is just the Steinberg morphism of G. Thus for arbitrary σ, we call St the Steinberg morphism of D and the fibers the Steinberg fibers of D.

By the previous proposition, any element $g \in D$ is of the form $t\sigma u$, where $t \in (T^\sigma)^0$ and u is a unipotent element in $Z^0(t\sigma_s)$. Moreover, $t\sigma_s$ is contained in the closure of the G-conjugacy class of g. Hence $\mathrm{St}(g) = \mathrm{St}(t\sigma_s)$. Notice that $t\sigma$ is quasi-semisimple in the sense of [St, Sect.9], i.e. the automorphism of G obtained by conjugation by $t\sigma_s$ will fix a Borel subgroup and a maximal torus thereof. As a consequence, the G-conjugacy class of $t\sigma_s$ in D is closed [Sp, II.1.15(f)]. We conclude that any Steinberg fiber of D is of the form

$$\bigcup_{g \in G} \bigcup_{u \text{ is unipotent in } Z_G(t\sigma_s)^0} g(t\sigma u)g^{-1}$$

for some $t \in (T^\sigma)^0$. It is known that $Z_G(t\sigma_s)^0$ is reductive and the set of unipotent elements in a reductive group is an irreducible variety. Thus

(a) each Steinberg fiber is irreducible.

Moreover, there are only finitely many unipotent conjugacy classes in a reductive group [L1]. Therefore

(b) each Steinberg fiber contains finitely many G-conjugacy classes.

Lemma 1.7. *Let* $\mathbf{a}$ *be a Steinberg fiber in* D *and* $J \subset I$ *with* $\sigma(J) = J$. *Then there exists finitely many* L_J*-conjugacy classes* $c_1, \cdots, c_m$ *in* $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$ *such that* $\{g; g \in N_{\tilde{G}}(P_J) \cap D, St(g) = \mathbf{a}\} = \sqcup_i c_i U_{P_J}$.

Let $l \in N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J)$ and $u \in U_{P_J}$. Then it is easy to see that l is contained in the closure of $\{tlut^{-1}; t \in (Z(L_J)^\sigma)^0\}$. Hence $St(l) = St(lu)$. In other words,

$$\begin{aligned} &\{g; g \in N_{\tilde{G}}(P_J) \cap D, St(g) = \mathbf{a}\} \\ &= \{lu; l \in N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D, u \in U_{P_J}, St(l) = \mathbf{a}\}. \end{aligned}$$

By [L2, Proposition 1.14], any quasi-semisimple element in D (resp. $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$) is G-conjugate (resp. L_J-conjugate) to $T_1\sigma$, where $T_1 = (T^\sigma)^0$. Notice that $\{t \in T_1; St(t\sigma) = \mathbf{a}\}$ is a finite

set. Hence there are only finitely many quasi-semisimple L_J-conjugacy classes in $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$ that are contained in $St_1(\mathbf{a})$. One can see that a L_J-conjugacy class c is contained in $St_1(\mathbf{a})$ if and only if the unique L_J-conjugacy class c' that is contained in c is also contained in $St_1(\mathbf{a})$. Then the lemma follows from Lemma 1.7 (b). $\square$

1.8. For $J \subset I$, set

$$Z_J = \{(P, P', gU_P); P, P' \in \mathcal{P}_J, g \in G, P' = {}^g P\},$$
$$Z'_J = \{(P, P', gH_P); P, P' \in \mathcal{P}_J, g \in G, P' = {}^g P\}$$

with the $G \times G$-action defined by

$$(g_1, g_2) \cdot (P, P', gU_P) = ({}^{g_2}P, {}^{g_1}P', g_1 g U_P g_2 1),$$
$$(g_1, g_2) \cdot (P, P', gH_P) = ({}^{g_2}P, {}^{g_1}P', g_1 g H_P g_2 1).$$

Set $h_J = (P_J, P_J, U_{P_J}) \in Z_J$ and $h'_J = (P_J, P_J, H_{P_J}) \in Z'_J$. By [L3, section 3], [H1, section 1] and the remark of [H2, Corollary 5.4], we have partitions

(a) $\qquad Z_J = \sqcup_{w \in {}^J W} Z_{J;w} \qquad$ and $\qquad Z'_J = \sqcup_{w \in {}^J W} Z'_{J;w},$

where $Z_{J;w} = G_\Delta \cdot (BwB, B) h_J$ and $Z'_{J;w} = G_\Delta \cdot (BwB, B) h'_J$. The subvarieties $Z_{J;w}$ (resp. $Z'_{J;w}$) are called G-stable pieces of Z_J (resp. Z'_J).

Fix $w \in {}^J W$. Let $K = I(J, id; w)$. Then by [L3, 3.14],
(b) there is a canonical bijection between the G_Δ-orbits on $Z_{J,w}$ and the L_K-conjugacy classes on wL_K.

By [L3, section 3], we have G-equivariant morphisms $pr : Z_{J;w} \to G/P_K$ and $pr' : Z'_{J;w} \to G/P_K$, where G acts diagonally on $Z_{J;w}$ and $Z'_{J;w}$ and acts in the natural way on G/P_K. Moreover, by [H1, Proposition 1.10],

(c) $\quad pr(z) = P_K$ if and only if $z = (pw, 1) \cdot h_J$ for some $p \in P_K$,
$\quad pr'(z) = P_K$ if and only if $z = (pw, 1) \cdot h'_J$ for some $p \in P_K$.

Also we have that the closure of any G-stable piece is a union of G-stable pieces.

(d) $\overline{Z_{J;w}} = \sqcup_{w' \in {}^J, w' \leqslant_{J,id} w} Z_{J,w'}$. See [H1, Proposition 4.6] and [H2, Proposition 5.8].

1.9. If $\sigma(J) = J$, then the action of $G \times G$ on Z_J and Z'_J can be extended in a natural way to an action of $\tilde{G} \times \tilde{G}$.

Now set

$$Ł_{J,D} = \{(z, g) \in Z_J \times D; (g, g) \cdot z = z\},$$
$$Ł'_{J,D} = \{(z, g) \in Z'_J \times D; (g, g) \cdot z = z\}.$$

For $w \in W^J$, set $Ł_{J,D;w} = \{(z, g) \in Ł_{J,D}; z \in Z_{J;w}\}$ and $Ł'_{J,D;w} = \{(z, g) \in Ł'_{J,D}; z \in Z'_{J;w}\}$.

1.10. Set $\tilde{P} = N_{\tilde{G}} P$ for any parabolic subgroup P of G. Define the action of P_J on $G \times \tilde{P}_J$ by $p \cdot (g, p') = (gp\imath, pp'p\imath)$. Let $G \times_{P_J} \tilde{P}_J$ be the quotient space. Then we may identify $G \times_{P_J} \tilde{P}_J$ with $\{(P, g); P \in \mathcal{P}_J, g \in \tilde{P}\}$ via $(g, p) \mapsto ({}^g P_J, gpg\imath)$.

Let c be a subvariety of $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$ that is stable under the conjugation action of L_J. Then cU_{P_J} and cH_{P_J} are stable under the conjugation action of P_J. So we may define $G \times_{P_J} cU_{P_J} \subset G \times_{P_J} cH_{P_J} \subset G \times_{P_J} \tilde{P}_J$.

Now set

$$\text{Ł}_{J,c} = \{(P, P', gU_P, h) \in \text{Ł}_{J,D}; (P, h), (P', h) \in G \times_{P_J} cU_{P_J}\},$$
$$\text{Ł}'_{J,c} = \{(P, P', gH_P, h) \in \text{Ł}'_{J,D}; (P, h), (P', h) \in G \times_{P_J} cH_{P_J}\}.$$

For $w \in W^J$, set $\text{Ł}_{J,c;w} = \text{Ł}_{J,D;w} \cap \text{Ł}_{J,c}$ and $\text{Ł}'_{J,c;w} = \text{Ł}'_{J,D;w} \cap \text{Ł}'_{J,c}$.

Lemma 1.11. *Let $w \in {}^J W$ and $K = I(J, id; w)$. Let c be a L_K-conjugacy class in $N_{\tilde{G}}(P_K) \cap N_{\tilde{G}}(L_K) \cap D$. Set $X_{J,c;w} = \{(z, g) \in L_{J,c;w}; pr(z) = P_K\}$ and $X'_{J,c;w} = \{(z, g) \in \text{Ł}'_{J,c;w}; pr'(z) = P_K\}$. Then*

(1) If $X_{J,c;w} \neq \emptyset$, then $X_{J,c;w}$ is of pure dimension $\dim(P_K)$.

(2) If $X'_{J,c;w} \neq \emptyset$, then $X'_{J,c;w}$ is of pure dimension

$$\dim(P_K) - \dim(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)})^\sigma.$$

Remark. *(1) If $X_{J,c;w} \neq \emptyset$ or $X'_{J,c;w} \neq \emptyset$, then as we will see in the proof, $\sigma \mathrm{Ad}(w)(z) = \mathrm{Ad}(w)\sigma(z)$ for $z \in Z^0(L_J)$. So we have that $\sigma Z^0(L_J) = Z^0(L_J)$ and $\sigma \mathrm{Ad}(w) Z^0(L_J) = \mathrm{Ad}(w) Z^0(L_J)$. Therefore $(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)})^\sigma$ is defined.*

(2) If $w(J) = J$, then $\mathrm{Ad}(w) Z^0(L_J) = Z^0(L_J)$ and

$$\dim(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)})^\sigma = 0.$$

If $w(J) \neq J$, then there exists $j \in J$ such that $w(j)$ is not spanned by the simple roots in J. Set $z_t = \prod_{j \notin J} \text{ø}_j(t)$ for $t \in \mathbf{k}^*$. Then $z_t \notin \mathrm{Ad}(w)\imath Z^0(L_J)$ for all $t \in \mathbf{k}^* - \{1\}$. Moreover, $\sigma \mathrm{Ad}(w)(z_t) = \mathrm{Ad}(w)\sigma(z_t) = \mathrm{Ad}(w)(z_t)$. Hence $\mathrm{Ad}(w) z_t \in (\mathrm{Ad}(w) Z^0(L_J))^\sigma$.

So $\dim(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)})^\sigma = 0$ if and only if $w(J) = J$.

We only prove part (1) here. Part (2) can be proved in a similar way.

Let $\bar{p} : P_K \to L_K$ be the projection map. By (c), $pr(z) = P_K$ if and only if $z = (pw, 1) \cdot h_J$ for some $p \in P_K$. Moreover, the morphism $f : \{z \in Z_{J;w}; pr(z) = P_K\} \to L_K$ defined by $(pw, 1) \cdot h_{J,D} \mapsto \bar{p}(p)$ is well-defined. Now consider the morphism $X_{J,c;w} \to L_K \times c$ which sends (z, g) to $(f(z), \bar{p}(g))$. We see that the image is contained in $\{(l, l') \in L_K \times c; lwl' = l'lw\}$, which is of pure dimension $\dim(L_K)$.

Let $l \in L_K$ and $l' \in c$ with $lwl' = l'lw$. Then the fiber Y over (l, l') is $\{(z, ul'); z \in (U_{P_K} lw, 1) \cdot h_J, u \in U_{P_K}, (ul', ul') \cdot z = z\}$. Define the action

of U_{P_K} on Y by $u_1 \cdot (z, ul') = ((u_1, u_1) \cdot z, u_1 ul' u_1 ^{-1})$. Then the projection map $Y \to (U_{P_K} lw, 1) \cdot h_J = (U_{P_K})_\Delta (lw, 1) \cdot h_J$ is U_{P_K}-equivariant for the diagonal U_{P_K}-action on $(U_{P_K} lw, 1) \cdot h_J$. Since U_{P_K} acts transitively on $(U_{P_K} lw, 1) \cdot h_J$, the projection map is a locally trivialy fibration with fibers isomorphic to $\{u \in U_{P_K}; (ulw, u) \cdot h_J = (lw, 1) \cdot h_J\}$. In particular, Y is of pure dimension $\dim(U_{P_K})$. Therefore $X_{J,c;w}$ is of pure dimenion $\dim(L_K) + \dim(U_{P_K}) = \dim(P_K)$. □

Proposition 1.12. *Let* $\mathbf{a}$ *be a Steinberg fiber of* $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$.

(1) If $Ł_{J,\mathbf{a};w} \neq \emptyset$*, then* $Ł_{J,\mathbf{a};w}$ *is of pure dimension* $\dim(G)$.

(2) If $Ł'_{J,\mathbf{a};w} \neq \emptyset$*, then* $Ł'_{J,\mathbf{a};w}$ *is of pure dimension*

$$\dim(G) - \dim\left(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)}\right)^\sigma.$$

We only prove part (1) here. Part (2) can be proved in a similar way.

Let $K = I(J, id; w)$. For $(z, g) \in X_{J,\mathbf{a};w}$, we have that ${}^g P_K = g \cdot pr(z) = pr\big((g, g) \cdot z\big) = pr(z) = P_K$. Hence $g \in P_K$. By lemma 1.7, $X_{J,\mathbf{a};w} = \sqcup_i X_{J,c_i;w}$ for finitely many L_K-conjugacy classes c_i. By lemma 1.11, $X_{J,\mathbf{a};w}$ is of pure dimension $\dim(P_K)$.

Let $\pi : Ł_{J,a;w} \to Z_{J;w}$ be the projection map. It is easy to see that π is G-equivariant for the diagonal G-action. Thus $pr \circ \pi : Ł_{J,\mathbf{a};w} \to G/P_K$ is also G-equivariant. Since G acts transitively on G/P_K, $pr \circ \pi$ is a locally trivial fibration with fibers isomorphism to $X_{J,\mathbf{a};w}$. Thus $Ł_{J,w;\mathbf{a}}$ is of pure dimension $\dim(G)$. □

Lemma 1.13. *Let* c *be a* L_J*-conjugacy class in* $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D$. *Set*

$$\mathcal{Z}_{J,c} = \{(P, P', g); (P, g), (P', g) \in G \times_{P_J} cU_{P_J}\},$$
$$\mathcal{Z}'_{J,c} = \{(P, P', g); (P, g), (P', g) \in G \times_{P_J} cH_{P_J}\}.$$

(1) Define the map $Ł_{J,c} \to \mathcal{Z}_{J,c}$ *by* $(P, P', kU_P, g) \mapsto (P, P', g)$*. If* $\mathcal{Z}_{J,c} \neq \emptyset$*, then the map is surjective and each fiber is of pure dimension* $\dim(L_J) - \dim(c)$.

(2) Define the map $Ł'_{J,c} \to \mathcal{Z}'_{J,c}$ *by* $(P, P', kH_P, g) \mapsto (P, P', g)$*. If* $\mathcal{Z}'_{J,c} \neq \emptyset$*, then the map is surjective and each fiber is of pure dimension* $\dim(L_J) - \dim(c) - \dim(Z(L_J)^\sigma)$.

We only prove part (1) here. Part (2) can be proved in the same way.

Let $(P, P', g) \in \mathcal{Z}_{J,c}$. Then there exists $k \in G$ such that $P' = {}^k P$. By definition, $kgk^{-1} U_{P'}$ and $gU_{P'}$ are P'-conjugate. Therefore, there exists $l \in P'$, such that $lkgk^{-1}l^{-1} \in gU_{P'}$. In other words, $(g, g) \cdot (P, P', lkU_P) = (P, P', lkU_P)$. So the map is surjective.

Assume that $(P, P', kU_P, g), (P, P', k'U_P, g) \in Ł_{J,c}$. Then $k^{-1}k' \in P$ and $(k^{-1}k') g U_P (k^{-1}k')^{-1} = gU_P$. Thus the fibers of the map $Ł_{J,c} \to \mathcal{Z}_{J,c}$ are isomorphic to $\{(lU_P; l \in P; lgU_P l^{-1} = gU_P\}$. Notice that $(P, g) \in G \times_{P_J} cU_{P_J}$. is of dimension $\dim(L_J) - \dim(c)$. □

Now combining Proposition 1.12 and Lemma 1.13, we have the following result.

Corollary 1.14. *Let c be a L_J-conjugacy class in $N_{\tilde{G}}(P_J) \cap N_{\tilde{G}}(L_J) \cap D'$. Then*

(1) $\dim(\mathcal{Z}_{J,c}) \leqslant \dim(G) - \dim(L_J) + \dim(c)$.

(2) $\dim(\mathcal{Z}'_{J,c}) \leqslant \dim(G) - \dim(L_J) + \dim(c) + \dim(Z(L_J)^\sigma)$. *More precisely, define* $\mathcal{Z}'_{J,c;w} = \{(P, P', g) \in \mathcal{Z}'_{J,c}; (P, P') \in G_\Delta \cdot (P_J, {}^w P_{J'})\}$ *for* $w \in {}^J W^{J'}$. *Then*

$$\dim(\mathcal{Z}'_{J,c;w}) \leqslant \dim(G) - \dim(L_J) + \dim(c) + \dim(Z(L_J)^\sigma) - \dim\Big(\frac{\mathrm{Ad}(w) Z^0(L_J)}{Z^0(L_J) \cap \mathrm{Ad}(w) Z^0(L_J)}\Big)^\sigma.$$

Remark. *Part (1) was first proved in* [L2, Proposition 4.2 (d)]. *By the remark of Lemma 1.11, part (2) is a stronger version of* [L2, Proposition 4.2(c)].

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